Math 206A Lecture 17 Notes

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1 Cauchy's Rigidity Theorem

1.1 Cauchy's Rigidity Theorem

Theorem 1.1 (Cauchy)**.** *Let $P, P' \subseteq \mathbb{R}^3$ be convex polytopes such that $\alpha(P) \simeq_\pi \alpha P'$, and for every 2-dimensional face F, $F \cong F'$ (up to rigid motion by $O_3(\mathbb{R}) \ltimes \mathbb{R}^3$). where $F' = \pi(F)$. Then $P \cong P'$.*

Example 1.1. Let P be a cube. If we have P' with 6 unit squares as faces, then $P \cong P'$. This is expected, since there is only 1 way to make a corner using 3 squares. But if P is an icosahedron, this is much less obvious.

We will make two mistakes in our proof, a small one and a medium-sized one.

Proof. Let $\Gamma = \Gamma(P) = \Gamma(P')$ be the graph of P. Γ is planar. Put pluses and minuses on the edges of the graph to indicate whether the dihedral angles of the faces meeting at those edges increase or decrease when going from P to P'. We know that the number of sign changes around each vertex is ≥ 4. This is our first mistake; since Cauchy's lemma's only applies to polygons, we cannot quite use it here. But a version of Cauchy's lemma for spherical polygons is true (and only relies on the spherical law of cosines).[1] Think of the cone extending from the vertex, and intersect it with a small sphere.

Let $M := \sum_{v \in V(P)} m_v$, where m_v is the number of sign changes around v. Then $M \geq 4|V|$. Let f_k be the number of k-sided faces in P. Then $|\mathcal{F}| = f_3 + f_4 + f_5 + \cdots$. We also know that $2|E| = 3f_3 + 4f_4 + 5f_5 + \cdots$. Subtracting these, we get

$$4|E| - 4|\mathcal{F}| = 2f_3 + 4f_4 + 6f_5 + \cdots,$$

and Euler's formula gives us $4|V| - 8 = 4|E| - 4|\mathcal{F}|$.

We claim that $M \leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + 8f_8 + \cdots$. In an n-gon, you can have at most n sign changes. We must also have an even number of sign changes. The trick is

[1] Back in 1900, spherical geometry was taught in high school. You might think that Professor Pak should know enough spherical geometry to know the spherical law of cosines, but he was, in fact, born after that time.

that we can count the sign changes around faces instead of vertices because if two edges are adjacent, then they bound the same face.

We now have that

$$2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + 8f_8 + \cdots \geq M > 4|V| - 8 = 2f_3 + 4f_4 + 6f_5 + \cdots ,$$

which is a contradiction. But we have made a medium-sized mistake; what if some angles have no sign change? In this case, draw the graphs but omit the edges with no sign changes. This still gives you a planar graph. What if the graph becomes disconnected? The 2 in Euler's formula ($|V| - |E| + |\mathcal{F}| = 2$) grows, which only works in favor of our inequality. $\square$

1.2 The story

Around the time of the French revolution, Legendre wanted to translate Euclid's Elements into French. It took him years, but he managed to translate it successfully. However, he discovered that there was a proof in the last section which was incorrect! He assigned Cauchy, his student, to prove the theorem and generalize it, and Cauchy, 19 at the time, did just that. There was a mistake in his proof of the arm lemma, but otherwise, Cauchy solved the problem. So in a sense, this theorem was proved as a consequence of the French revolution.